Charting the Space of Superconformal Field Theories

The Power of Quantum Consistency

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Related to work in collaboration with
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Quantum Field Theory
"Theory of quantum fields" (duh!)

$$\int \prod_x d\varphi(x)\, e^{-\frac{S[\varphi]}{\hbar}}$$

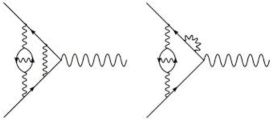
Infinite-dimensional integral handled by

- Introducing a cut-off (e.g., $x \in$ Lattice)
- Renormalization theory

Mathematicians may get a little nervous, but we think we know what we are doing...

$$S = \int d^D x \, \mathcal{L} \, , \quad \mathcal{L} = \text{quadratic} + g^2 \varphi^4 + \dots$$

- "Easy" when $g \to 0$. Perturbative expansion:



$$\text{Rescaling } \; \varphi \to \varphi/g \text{ gives } e^{-\frac{S}{g^2 \hbar}}$$

$g \to 0$ equivalent to classical limit $\hbar \to 0$

- Hard for large g. Lattice simulations, ...

"QFT is about Fields and Lagrangians then . . ." But is it?

Discoveries in supersymmetric field theories in various spacetime dimensions are challenging the traditional framework.

Supersymmetry allows for analytic control, but the conceptual lessons are likely to be general.

Inadequacy of “fields” I: Dualities

In happy cases, as $g \to \infty$ an equivalent dual description emerges.

Pair of dual theories

$$\mathcal{T}[\varphi_i; g] \Leftrightarrow \mathcal{T}'[\varphi'_i; g'], \quad g' = \frac{1}{g}$$

$\mathcal{T}$ and $\mathcal{T}'$ different classical limits of the same quantum theory.

φ and φ' not fundamental objects.

Some QFTs are even dual to quantum gravity theories
(in higher spacetime dimensions)!

All that is solid melts into air
Fields, gauge symmetries, spacetime itself...not fundamental?

Paradigm: S-duality of $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions.
Maximally symmetric cousin of QCD.

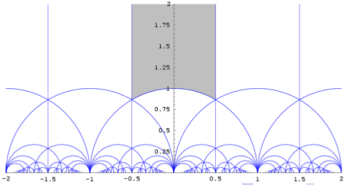
Complexified gauge coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \in$ upper-half plane H.

$SL(2, \mathbb{Z})$ duality (generalizing electric-magnetic duality)

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

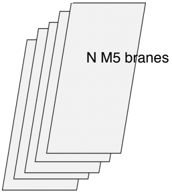
with a, b, c, d integers and $ad - bc = 1$.
No path-integral derivation *remotely* in sight...

"Theory space"
$\tau \in H/SL(2, \mathbb{Z})$



Inadequacy of "fields" II: non-Lagrangian QFTs

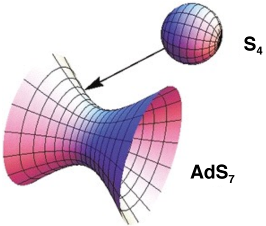
$d = 6$ maximally SUSY theory, known as the (2, 0) theory.



$(2,0)_N$ theory governs low-energy fluctuations of N five-branes in M-theory

Discrete parameter N, but no continuous coupling.
For finite N, intrinsically quantum.

As $N \to \infty$
classical 11d supergravity
on $AdS_7 \times S^4$



Non-Lagrangian theories in $d = 4$ Gaiotto 2008

A sequence of theories with half-maximal SUSY ($\mathcal{N} = 2$):

"Trinions" T_N

$SU(N) \times SU(N) \times SU(N)$ global symmetry.

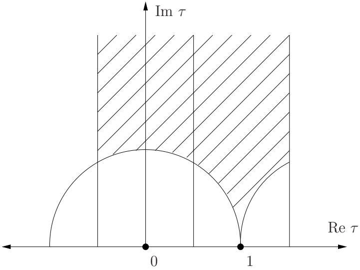
A new kind of strongly coupled "matter".
These new creatures can be coupled to gauge fields and used as building blocks for more complex theories, much like ordinary Lagrangian matter.

If this sounds contrived...

Dualities 2.0

Lagrangian theories are not closed under dualities!

"$\mathcal{N} = 2$ version of QCD", $SU(3)_c$ gauge group with $N_f = 6$ quarks.



After accounting for "standard" dualities, still an infinite coupling point at $\tau = 1$.

Novel dual description:
T_3 theory with weakly-coupled $SU(2)_c$ gauge fields!
Argyres Seiberg 2007

Far from isolated curiosities, the T_N theories are fundamental to piece together the $\mathcal{N} = 2$ landscape.

Only a measure zero set of the known $\mathcal{N} = 2$ theories have a Lagrangian description...

Beyond Lagrangian field theory

In the following, I will advocate an abstract algebraic viewpoint to study superconformal field theories.

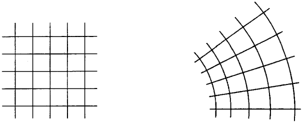
These theories are extremely "rigid".
Quite possibly, uniquely fixed by quantum consistency alone.

I will explore two notions of consistency:

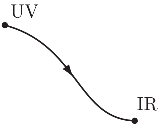
- Consistency of the abstract algebra of operators in a given theory
- Consistency of the duality relations in the space of theories.

Conformal symmetry

Physics simplifies when intrinsic mass scales can be neglected: large/low energy regimes of QFTs and statistical systems near T_c.
Scale invariance. "Generically" enhanced to conformal invariance.
A conformal transformation acts *locally* as rotation and dilatation:



CFTs are signposts in the space of QFTs.



(Conjecture) Generic behavior of (unitary) QFT: an RG flow between two CFTs.
$\mathrm{DOF}_{UV} > \mathrm{DOF}_{IR}$

Abstract CFT

A CFT is defined by a set of local operators $\{\mathcal{O}_k(x)\}$ and their correlators

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \,.$$

Organize operators in conformal families: primary + derivatives

$$\{\mathcal{O}_i, \partial\mathcal{O}_i \,, \partial^2\mathcal{O}_i \,, \dots \}$$

Scaling dimensions Δ_i: $\langle \mathcal{O}_i(x)\mathcal{O}_i(y) \rangle = |x-y|^{-2\Delta_i}$

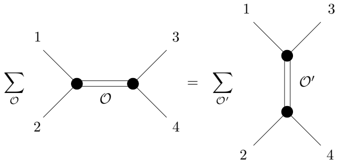
Convergent Operator Product Expansion

$$\text{OPE}: \quad \mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k c_{ijk}\, x^{\Delta_k - \Delta_i - \Delta_j} \left(\mathcal{O}_k(0) + \dots \right) \,.$$

c_{ijk}: 3pt couplings of primaries. *Dots* fixed by conformal invariance.

Conformal bootstrap Polyakov, Ferrara Gatto Grillo '70s

Decompose $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$ in two ways



E.g., for four identical external operators $\varphi(x_i)$

$$\sum_{\mathcal{O}_j} c^2_{\varphi\varphi j} G_{\mathcal{O}_j}(u,v) = \sum_{\mathcal{O}_j} c^2_{\varphi\varphi j} G_{\mathcal{O}_j}(v,u)$$

$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ and $G_{\mathcal{O}_j}(u,v)$ resums contribution of $\{\partial^n \mathcal{O}_j\}$.

Vastly over-constrained system of equations for $\{\Delta_i, c_{ijk}\}$.

Solve or carve?

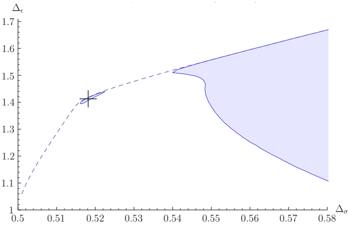
- Traditional bootstrap
  Ideally, given some minimal physical input, we'd just find the (unique?) analytic solution of the bootstrap equations.

  Famous success story in $d = 2$ Belavin Polyakov Zamolodchikov 1984: conformal symmetry $z \to f(z)$, exact solution of many models.

- Modern bootstrap Rattazzi Rychkov Tonni Vichi 2008

  Crossing + unitarity $\Rightarrow$ inequalities for $\{\Delta_i, c_{ijk}\}$.

  Carve out numerically the space of CFT data. Works in any d.



Kos Poland Simmons-Duffin 2014

Exact bootstrap in SUSY theories

Beem Lemos Liendo Peelaers LR van Rees 2013

Today, first strategy.

Focus on CFTs with $\mathcal{N} = 2$ SUSY in $d = 4$ or $(2, 0)$ SUSY in $d = 6$.

Special supersymmetric operators, "$Q\mathcal{O}_{\rm BPS} = 0$".

Truncation of operator algebra to $\{\mathcal{O}_{\rm BPS}\}$ that lie on a plane $\mathbb{C}_{[z,\bar{z}]} \subset \mathbb{R}^d$

$$\mathcal{O}_i(z)\mathcal{O}_j(w) \sim \sum_k \frac{c_{ij}^k \, \mathcal{O}_k(w)}{(z-w)^{h_i+h_j-h_k}}$$

Even simpler than $2d$ CFT: z but no $\bar{z}$ dependence:

known as a chiral algebra.

BPS bootstrap of $(2,0)_N$ Beem LR van Rees 2014

Abstract approach is all we have for $(2,0)$ theory.

Using minimal input (from string/M-theory) on BPS spectrum, able to solve meromorphic bootstrap equations.

Chiral algebra of BPS operators = Famous W_N algebra

Exact 3-point functions of for any N.
For $N \to \infty$,

$$c_{k_1 k_2 k_3} = \frac{2^{2\alpha-2}}{(\pi N)^{\frac{3}{2}}} \Gamma\left(\frac{\alpha}{2}\right) \left(\frac{\Gamma\left(\frac{k_{123}+1}{2}\right)\Gamma\left(\frac{k_{231}+1}{2}\right)\Gamma\left(\frac{k_{312}+1}{2}\right)}{\sqrt{\Gamma(2k_1-1)\Gamma(2k_2-1)\Gamma(2k_3-1)}} \right) .$$

Striking agreement with supergravity on $AdS_7 \times S^4$.
One recovers non-linear SUGRA purely from algebraic consistency.

In principle, $1/N$ corrections $\Rightarrow$ quantum M-theory corrections.

BPS bootstrap of T_N

Beem Peelaers LR van Rees, Lemos Peelaers 2014

Intricate BPS spectrum: generators in Λ^k irrep of three $SU(N)$'s.

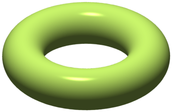
- T_2 chiral algebra: free symplectic bosons
- T_3 chiral algebra: E_6 current algebra at level $k = -3$.
- T_4 chiral algebra: a novel, exceptional W-algebra.
  Miraculous solution to crossing. New dynamical information.
- $T_{N \geqslant 5}$: still unsolved.

$6 = 4 + 2$ Witten 1995

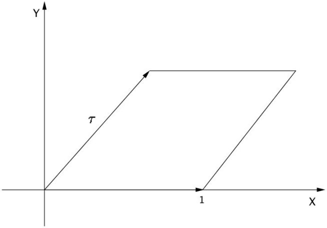
Put $(2,0)_N$ on $\mathbb{R}^4 \times T^2$. Flow to the IR

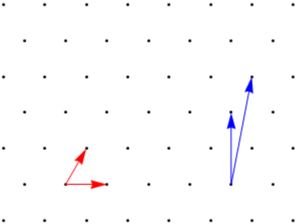
$\Downarrow$

$SU(N)$ $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R}^4$
with coupling $\tau \equiv$ modular parameter of T^2.



This picture "explains" S-duality.

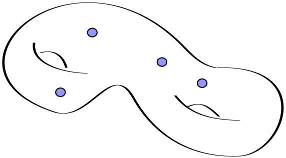




$\tau' = \frac{a\tau+b}{c\tau+d}$ defines the same T^2.

$6 = 4 + 2$: class $\mathcal{S}$(ix) theories Gaiotto 2008

Put $(2,0)_N$ on $\mathbb{R}^4 \times \mathcal{C}$.
$\mathcal{C} \equiv$ Riemann surface with punctures.
$\Downarrow$
$\mathcal{N} = 2$ SUSY CFT on $\mathbb{R}^4$.

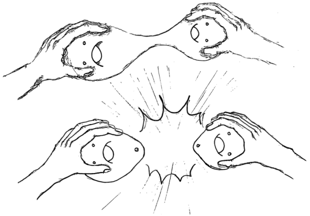


$4d$ **SCFT** $\mathcal{T}[\mathcal{C}]$	$2d$ **data on** $\mathcal{C}$
Gauge couplings $\{\tau_i\}$	Complex moduli of $\mathcal{C}$
$SU(N)$ gauge group	cylinder
T_N	Three-punctured sphere (trinion)
Global symmetry	Puncture
Generalized S-duality	Modular transformation of $\mathcal{C}$

Theory space interpreted as a "real" geometric space, the surface $\mathcal{C}$.

Consistency conditions in theory space

Degeneration of $\mathcal{C} \Rightarrow$ Theory $\mathcal{T}[\mathcal{C}]$ splits into decoupled theories.
Very powerful constraint.



[Roy Sato's drawing, from Tachikawa's webpage]

Consistency in theory space:
Gluing of surfaces translates into gluing rules for physical observables.

Enough to fix several SUSY observables, provided minimal physical input.

Example: twisted $S^3 \times S^1$ partition function

Witten index, encoding the SUSY spectrum:

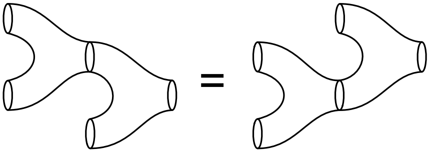
$$\mathcal{I}(q,p,t;a_i) = \mathrm{Tr}(-1)^F \, p^{j_{12}-r} \, q^{j_{34}-r} \, t^{R+r} \prod_i a_i{}^{f_i} \, .$$

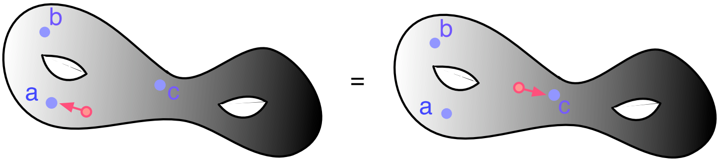
(p,q,t) geometric parameters of the twist.
a_i fugacities associated to the flavor symmetry of $\mathcal{T}$.

For a Lagrangian theory, $\mathcal{I}[\mathcal{T}]=$ elliptic hypergeometric integral.
Basic ingredient is Elliptic gamma function

$$\mathcal{I} = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}t^{-\frac{1}{2}}}{1-p^m q^n t^{\frac{1}{2}}} \equiv \Gamma(t^{\frac{1}{2}};p,q) \, .$$

Consistency in theory space fixes $\mathcal{I}[\mathcal{T}]$ uniquely for all $\mathcal{T}[\mathcal{C}]$!
Computed by a Topological QFT living on $\mathcal{C}$.
Gadde Gaiotto Razamat LR Yan





$$\mathfrak{S}_{(r,s)}(\mathbf{a})\,\mathcal{I}_{\mathcal{C}}(\mathbf{a},\mathbf{b},\mathbf{c}\cdots) = \mathfrak{S}_{(r,s)}(\mathbf{c})\,\mathcal{I}_{\mathcal{C}}(\mathbf{a},\mathbf{b},\mathbf{c},\cdots)$$

Difference equation of the elliptic RS model.

Outlook

- Still exploring very rich algebraic structure of BPS observables. Goldmine of interesting mathematical physics.

- Numerical bootstrap for non-BPS observables.

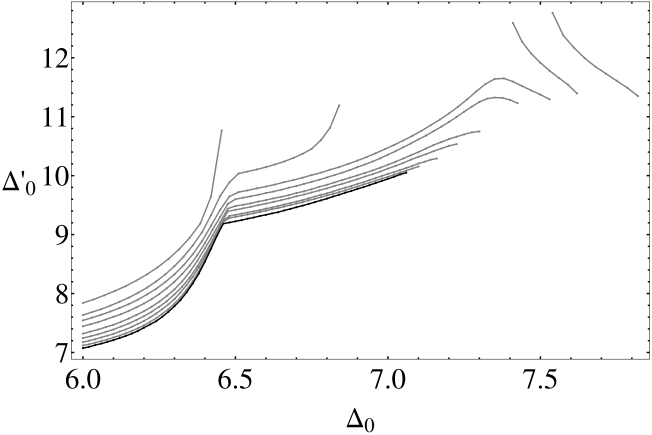
  Cornering $(2,0)$?
  Beem Lemos LR van Rees, to appear

  Broad exploration of $\mathcal{N}=2$ landscape in $d=4$
  Beem Lemos Liendo LR van Rees, 2015

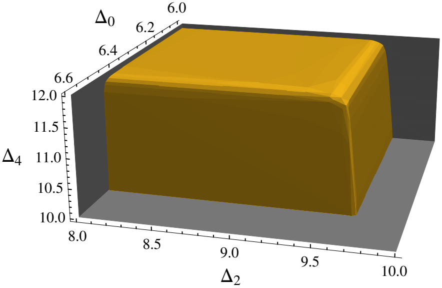
- Conceptual and analytical questions on the bootstrap.

Numerical bootstrap of $(2,0)$ Theory

Beem Lemos LR van Rees, to appear



Cornering the $(2,0)_2$ theory!



Exclusion region for $(2,0)_2$.

The corner values conjectured to be the actual leading-twist dimensions $(\Delta_0, \Delta_2, \Delta_4)$ of the theory.

Conclusions

We're still learning what QFT is.

- From exploration of superconformal theories,
  new insights into the meaning of QFT.
- New practical tools,
  such as the numerical bootstrap and the exact BPS bootstrap.
- New mathematics.

I've emphasized two heuristic principles:

- "Bootstrap" approach:
  Use general principles, as opposed to detailed dynamical models.
- Enlarge the view to the whole space of QFTs.

(From the Salt Lake Tribune)

Pull yourself up from the mud of theory space!

Thank you!