Amplitude method studies of effective field theories

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Outline:

- Role model: gluon amplitudes
- Effective Field Theories
- From NLSM to Periodic Table of Scalar Theories
- Further avenues
- New soft theorem
- Summary

Introduction: amplitudes

Objective of amplitude community:

Study a priori known objects from different perspective

Example in mind: gluon amplitudes

- 1986: Parke and Taylor calculated 6-point gluon-scattering
- simplification: tree-level, no-fermions
- final result: extremely simple
- other way of calculation?

Example: gluon amplitudes

At tree level:

 $\bullet\ colour\ ordering\ \rightarrow\ stripped\ amplitude$

$$M^{a_1\ldots a_n}(p_1,\ldots p_n) = \sum_{\sigma/Z_n} \operatorname{Tr}(t^{a_{\sigma(1)}}\ldots t^{a_{\sigma(n)}}) M_{\sigma}(p_1,\ldots,p_n)$$

•
$$M_{\sigma}(p_{\sigma(1)},\ldots,p_{\sigma(n)})=M(p_1,\ldots,p_n)\equiv M(1,2,\ldots,n)$$

- propagators \Rightarrow the only poles of M_σ
- thanks to ordering the only possible poles are:

$$P_{ij}^2 = (p_i + p_{i+1} + \ldots + p_{j-1} + p_j)^2$$

Pole structure

Weinberg's theorem (one particle unitarity): on the factorization channel

$$\lim_{P_{1j}^2 \to 0} M(1, 2, \dots, n) = \sum_{h_l} M_L(1, 2 \dots, j, l) \times \frac{1}{P_{1j}^2} \times M_R(l, j+1, \dots, n)$$



BCFW relations, preliminaries

[Britto, Cachazo, Feng, Witten '05]

Reconstruct the amplitude from its poles (in complex plane)

• shift in two external momenta

$$p_i \rightarrow p_i + zq$$
, $p_j \rightarrow p_j - zq$

• keep p_i and p_j on-shell, i.e.

$$q^2 = q \cdot p_i = q \cdot p_j = 0$$

- amplitude becomes a meromorphic function A(z)
- only simple poles coming from propagators $P_{ab}(z)$
- original function is A(0)



$$\frac{1}{2\pi i} \int \frac{dz}{z} A(z) = A(0) + \sum_{k} \frac{\operatorname{Res}(A, z_{k})}{z_{k}}$$



$$\mathbf{0} = \frac{1}{2\pi \mathrm{i}} \int \frac{dz}{z} A(z) = A(0) + \sum_{k} \frac{\mathrm{Res}(A, z_{k})}{z_{k}}$$

If A(z) vanishes for $z \to \infty$

$$A = A(0) = -\sum_{k} \frac{\operatorname{Res}(A, z_k)}{z_k}$$

BCFW relations

 $P_{ab}^2(z) = 0$ if one and only one *i* (or *j*) in $(a, a + 1, \dots, b)$.

Suppose $i \in (a, \ldots, b) \not\ni j$

$$\frac{P_{ab}^{2}(z)}{P_{ab}^{2}(z)} = (p_{a} + \ldots + p_{i-1} + p_{i} + zq + p_{i+1} + \ldots + p_{b})^{2} =$$
$$= P_{ab}^{2} + 2q \cdot P_{ab}z = 0$$

solution

$$z_{ab} = -rac{P_{ab}^2}{2(q \cdot P_{ab})} \qquad \Rightarrow \qquad P_{ab}^2(z) = -rac{P_{ab}^2}{z_{ab}}(z - z_{ab})$$

Thus

$$\operatorname{Res}(A, z_{ab}) = \sum_{s} A_{L}^{-s}(z_{ab}) \times \frac{-z_{ab}}{P_{ab}^{2}} \times A_{R}^{s}(z_{ab})$$

and for allowed helicities it factorizes into two subamplitudes

BCFW relations

Using Cauchy's formula, we have finally as a result

$$\mathcal{A} = \sum_{k,s} \mathcal{A}_L^{-s_k}(z_k) rac{1}{P_k^2} \mathcal{A}_R^{s_k}(z_k)$$

- based on two-line shift (convenient choice: adjacent *i*,*j*)
- recursive formula (down to 3-pt amplitudes)
- number of terms small = number of factorization channels
- all ingredients are on shell

BCFW Example: gluon amplitudes

od diagrams for $\mathit{n}\text{-}\mathsf{body}$ gluon scatterings at tree level

п	3	4	5	6	7	8
<pre># diagrams (inc.crossing)</pre>	1	4	25	220	2485	34300
# diagrams (col.ordered)		3	10	38	154	654
# BCFW terms	-	1	2	3	6	20

[C.Cheung: TASI Lectures '17] [KK, Novotny, Trnka '13]

BCFW recursion relations: problems

We have assumed that

A(z)
ightarrow 0, for $z
ightarrow \infty$

More generally we have to include a boundary term in Cauchy's theorem.

This is intuitively clear: we can formally use the derived BCFW recursion relations to obtain any higher n amplitude starting with the leading interaction. But this does not have to be the correct answer.

BCFW recursion relations: problems example: scalar-QED



Due to the power-counting the boundary term is proportional to

$$B\sim 2e^2-\lambda$$

In order to eliminate the boundary term we have to set $\lambda = 2e^2$, then the original BCFW works.

I.e. we needed some further information (e.g. supersymmetry) to determine the λ piece.

Effective field theories

Effective field theories: general picture

Now we have infinitely many unfixed " λ " terms. Schematically

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \lambda_4 (\partial^{m_4} \phi)^4 + \lambda_6 (\partial^{m_6} \phi)^6 + \dots$$

Example: 6pt scattering, Feynman diagrams



Corresponding amplitude:

$$\mathcal{M}_6 = \sum_{I=poles} \lambda_4^2 \frac{\cdots}{P_I} + \frac{\lambda_6}{(\cdots)}$$

 λ_6 part: not fixed by the pole behaviour.

Task: to find a condition in order to link these two terms

Effective field theories: introduction

Usual steps:

```
\begin{array}{l} \mathsf{Symmetry} \to \mathsf{Lagrangian} \to \mathsf{Amplitudes} \to \mathsf{physical} \ \mathsf{quantities} \\ & (\mathsf{cross-section}, \ \mathsf{masses}, \\ & \mathsf{decay} \ \mathsf{constants}, \ \ldots) \end{array}
```

In our work – opposite direction:

 $\textbf{Amplitudes} \rightarrow \textbf{Lagrangian} \rightarrow \textbf{Symmetry}$

Our aim: classification of interesting EFTs

works done in collaborations with Clifford Cheung, Jiri Novotny, Chia-Hsien Shen, Jaroslav Trnka and Congkao Wen

Effective field theories: scalar theories

As simple as possible: a spin-0 massless degree of freedom with a three-point interaction.

General formula for three-particle amplitude

$$\mathcal{A}(1^{h_1}2^{h_2}3^{h_3}) = \left\{ \begin{array}{ll} \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 23 \rangle^{h_1 - h_2 - h_3} \langle 31 \rangle^{h_2 - h_3 - h_1}, & \Sigma h_i \leq 0\\ [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2}, & \Sigma h_i \geq 0 \end{array} \right.$$

Used a spinor-helicity notation, e.g. $p_i \cdot p_j \sim \langle ij \rangle [ij]$

For scalars $(h_i = 0)$ this is a constant - corresponding to $\mathcal{L}_{int} = \lambda \phi^3$. All derivatives can be removed by equations of motions (boxes)

$$\mathcal{L}_{int} = (\partial_{lpha} \dots \partial_{\omega} \phi) (\partial^{lpha} \dots \partial^{\omega} \phi) \phi \quad o \quad \mathcal{L}_{int} = (\Box \phi) (\dots)$$

Effective field theories: scalar theories

We start with (*m* counts number of derivatives)

$$\mathcal{L}_{int} = \lambda_4 \partial^m \phi^4$$

n.b. we want to connect this four-point vertex with the 6-point contact terms



This rules out again no-derivative terms, as the powercounting dictates:

$$\partial^m imes rac{1}{\partial^2} imes \partial^m \quad o \quad \partial^{2m-2} \phi^6$$

Simplest example: two derivatives, single scalar

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \lambda_4 \partial^2 \phi^4 + \lambda_6 \partial^2 \phi^6 + \dots$$

How to connect λ_4 and λ_6 ?

Well Lagrangian, an infinite series, looks complicated, but it is not the case. It represents a free theory:

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \underbrace{(1 + \lambda_{4} \phi^{2} + \ldots)}_{F(\phi)}$$

 $F(\phi)$ can be removed by a field redefinition

Non-trivial simplest example:

more derivatives

• more flavours
$$(\phi \rightarrow \phi_1, \phi_2, \ldots)$$

More flavours

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^{i} \partial^{\mu} \phi^{i} + \lambda_{ijkl} \partial_{\mu} \phi^{i} \partial^{\mu} \phi^{j} \phi^{k} \phi^{l} + \lambda_{i_{1} \dots I_{6}} \partial_{\mu} \phi^{i_{1}} \partial^{\mu} \phi^{i_{2}} \phi^{i_{3}} \dots \phi^{i_{6}} + \dots$$

- Can be used for systematic studies of two species, three species, etc.
- Very complicated generally
- Assume some simplification, organize using a group structure

$$\phi = \phi^a T^a$$

• motivated by the 'gluon case': flavour ordering [KK,Novotny,Trnka '13]

$$A^{a_1\dots a_n} = \sum_{perm} \operatorname{Tr}(T^{a_1}\dots T^{a_n})A(p_1,\dots p_n)$$

More flavours: stripped amplitude

first non-trivial case 6pt scattering:



power-counting is ok:

$$\lambda_4^2 p^2 \frac{1}{p^2} p^2 + \frac{\lambda_6}{p^2} p^2$$

in order to combine the pole and contact term we need to consider some limit. The most natural candidate: We will demand soft limit, i.e.

$$A \rightarrow 0$$
, for $p \rightarrow 0$

 $\Rightarrow \lambda_4^2 \sim \lambda_6$

Standard direction(s)

Assuming the shift symmetry

$$\phi^{a} \to \phi^{a} + \epsilon^{a}$$

 $\Rightarrow \mathsf{Noether}\ \mathsf{current}$

$$A^{a}_{\mu} = \frac{\delta \mathcal{L}}{\delta \partial^{\mu} \phi^{a}}$$

 $\Rightarrow \mathsf{Ward} \ \mathsf{identity} \Rightarrow \mathsf{LSZ}$

$$\langle 0|A^{a}_{\mu}(x)|\phi^{b}(p)
angle = iF\delta^{ab}p_{\mu}e^{-ipx}$$

 \Rightarrow Adler zero

$$\lim_{p\to 0} \langle f|i+\phi^a(p)\rangle = 0$$

 \Rightarrow CCWZ: non-linear sigma model

$$\mathcal{L} = rac{F^2}{2} \operatorname{Tr}(\partial_{\mu} U^{\dagger} \partial^{\mu} U), \qquad U = \mathrm{e}^{rac{i}{F} \phi^{a} T^{a}}$$

[Weinber'66], [Ian Low '14-'15]

Natural classification: σ and ρ

Soft limit of one external leg of the tree-level amplitude

$$A(tp_1, p_2, \ldots, p_n) = \mathcal{O}(t^{\sigma}), \quad \text{as} \quad tp_1 \to 0$$

Interaction term

$$\mathcal{L} = \partial^m \phi^n$$

Then another natural parameter is (counts the homogeneity)



so these two diagrams can mix: $p^{2m-2} \sim p^{\widetilde{m}}$

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Non-trivial cases

for:
$$\mathcal{L} = \partial^m \phi^n$$
: $m < \sigma n$

or

$$\sigma > \frac{(n-2)\rho + 2}{n}$$

i.e.

ρ	σ at least		
0	1		
1	2		
2	2		
3	3		

i.e. non-trivial regime for $\rho \leq \sigma$

First case: $\rho = 0$ (i.e. two derivatives)

Schematically for single scalar case

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \sum_i \lambda_4^i (\partial^2 \phi^4) + \sum_i \lambda_6^i (\partial^2 \phi^6) + \dots$$

similarly for multi-flavour (ϕ_i : ϕ_1, ϕ_2, \ldots). non-trivial case

$$\sigma = 1$$

Outcome:

- single scalar: free theory
- multiple scalars (with flavour-ordering): non-linear sigma model
- n.b. it represents a generalization of [Susskind, Frye '70], [Ellis, Renner '70]

1. focus on the lowest combination and fix the form:

$$\mathcal{L}_{int} = c_2 (\partial \phi \cdot \partial \phi)^2 + c_3 (\partial \phi \cdot \partial \phi)^3 \qquad \text{condition: } c_3 = 4c_2^4$$

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3. ansatz of the form

$$c_n(\partial\phi\cdot\partial\phi)^n+c_{n+1}(\partial\phi\cdot\partial\phi)^n\partial\phi\cdot\partial\phi$$

4. in order to cancel: $2(n+1)c_{n+1} = (2n-1)c_n$ i.e. $c_1 = \frac{1}{2} \Rightarrow c_2 = \frac{1}{8}, c_3 = \frac{1}{16}, c_4 = \frac{5}{128}, \dots$

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$$\mathcal{L} = -\sqrt{1 - (\partial \phi \cdot \partial \phi)}$$

This theory known as a scalar part of the Dirac-Born-Infeld [1934] – DBI action

Scalar field can be seen as a fluctuation of a 4-dim brane in five-dim Minkowski space



Remark: soft limit and symmetry are "equivalent"

Third case: $\rho = 2$, $\sigma = 2$ (double soft limit)

Similarly to previous case we will arrive to a unique solution: the Galileon Lagrangian

$$\mathcal{L} = \sum_{n=1}^{d+1} d_n \phi \mathcal{L}_{n-1}^{\mathrm{der}}$$

$$\mathcal{L}_n^{\mathrm{der}} = \varepsilon^{\mu_1 \dots \mu_d} \varepsilon^{\nu_1 \dots \nu_d} \prod_{i=1}^n \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n+1}^d \eta_{\mu_j \nu_j} = -(d-n)! \det \left\{ \partial^{\nu_i} \partial_{\nu_j} \phi \right\}.$$

It possesses the Galilean shift symmetry

$$\phi \rightarrow \phi + a + b_{\mu} x^{\mu}$$

(leads to EoM of second-order in field derivatives)

Surprise: $\rho = 2$, $\sigma = 3$ (enhanced soft limit)

• general galileon: three parameters (in 4D)

• only two relevant (due to dualities [de Rham, Keltner, Tolley '14] [KK, Novotny '14])

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- we demanded $\mathcal{O}(p^3)$ behaviour
- we have verified: possible up to very high-pt order
- suggested new theory: special galileon [Cheung,KK,Novotny,Trnka 1412.4095]

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- suggested new theory: special galileon [Cheung,KK,Novotny,Trnka 1412.4095]
- symmetry explanation: hidden symmetry [K. Hinterbichler and A. Joyce 1501.07600]

$$\phi \to \phi + s_{\mu\nu} x^{\mu} x^{\nu} - 12\lambda_4 s^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$$

New recursion for effective theories [Cheung, KK, Novotny, Shen, Trnka 2015]

The high energy behaviour forbids a naive Cauchy formula

$$A(z)
eq 0 \qquad ext{for} \quad z o\infty$$

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The high energy behaviour forbids a naive Cauchy formula

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ightarrow\infty$

Can we instead use the soft limit directly? yes! The standard BCFW not applicable, we propose a special shift:

 $p_i
ightarrow p_i(1-za_i)$ on all external legs

This leads to a modified Cauchy formula

$$\oint \frac{dz}{z} \frac{A(z)}{\prod_i (1-a_i z)^{\sigma}} = 0$$

note there are no poles at $z = 1/a_i$ (by construction). Now we can continue in analogy with BCFW

Further avenues

• similarly for vector EFT:

$$\mathcal{L}_{\mathrm{BI}} = 1 - \sqrt{(-1)^{D-1} \mathrm{det}(\eta_{\mu\nu} + F_{\mu\nu})},$$

(see [Cheung, KK, Novotny, Shen, Trnka, Wen '18])

- so far avoided the fermionic degrees of freedom (see e.g. Elvang et al.'18)
- higher orders in NLSM (see [Bijnens,KK,Sjö 2019])
- multiple flavours especially without flavour ordering
- only two-flavour case fully classified
- preliminary study of the mixed scalar-vector case (Galileon-BI): more promising than the pure Galileon-like BI
- spin-2: similar to Galileon-like studies no exceptional candidate
- non-abelian Born-Infeld
- non-zero masses (technically possible)
- loop corrections focused on the exceptional theories

BUT...

[KK, Novotny, Shifman, Trnka 2019 and in prep.]

- it seems we have a powerful method to classify effective field theories
- more efficient than the standard group oriented methods: spontaneous symmetry breaking, (non-)compact groups, (semi-)simple, CCWZ construction ... → complicated monomial structure, where equivalence is not transparent
- e.g. for two flavours two-derivative counting: only one non-trivial theory: O(3)/O(2)
- more problematic for three flavours finished (work in progress with J.Bijnens): again only O(4)/O(3) and combinations of O(3)/O(2) plus one free scalar
- but what about completely broken O(3):

SU(2)/1

SU(2)/1

- it describes three GBs
- CCWZ construction
- ullet \Rightarrow Lagrangian
- it is neither equivalent to O(4)/O(3) nor to O(3)/O(2), nor any their flavour combinations
- on top of it: the amplitudes don't have adler zero!
- What have we missed?

General discussion

- answer is then easy we missed "non-zero" Adler zero
- beyond the scope of our classification
- so our method is not that general
- can we extend it?

Adler zero: textbook derivation

GB couples to the associated Noether current

$$\langle 0|J^{\mu}(x)|\phi(p)
angle = -ip^{\mu}F\mathrm{e}^{-ip.x}$$

For the process $i \to f + \phi(p)$ we have:

$$\langle f|J^{\mu}(0)|i\rangle = F rac{p^{\mu}}{p^2} \mathcal{A}(f+\phi(p),i) + R^{\mu}(p)$$

The current conservation $p_{\mu}\langle f|J^{\mu}(0)|i
angle=0$ yields

$$\mathcal{A}(f+\phi(p),i)=-rac{1}{F}p_{\mu}R^{\mu}(p)$$

and thus finally

$$\lim_{p\to 0} \mathcal{A}(f+\phi(p),i) = 0$$

if R(p) regular in the limit.

Adler zero: loophole

When

$$\lim_{p\to 0} p_{\mu} R^{\mu} \neq 0 \qquad ?$$



• there are cubic vertices





SU(2)/1: new soft theorem

- n.b.:three GBs, can be rotated to: two charged ϕ^\pm and one neutral $\phi.$
- Simplification: charge conservation + shift symmetry in the neutral mode.
- standard Adler zero for the neutral mode
- we can focus only on the ϕ^+ shift. Ansatz:

$$\lim_{p_1 \to 0} \phi^+(p_1) \dots \phi^+(p_n) \phi^-(q_1) \dots \phi^-(q_n) \phi(k_1) \dots \phi(k_m) =$$

$$x \sum_{i=1}^m \phi^+(\mathbf{k}_i) \dots \phi^+(p_n) \phi^-(q_1) \dots \phi^-(q_n) \phi(k_1) \stackrel{\text{\tiny ϕ}(\mathbf{k}_i)}{\dots} \phi(k_m)$$

$$+y \sum_{i=1}^n \stackrel{\text{\tiny ϕ}^+(\mathbf{p}_1)}{\dots} \dots \phi^+(p_n) \phi^-(q_1) \stackrel{\text{\tiny ϕ}(\mathbf{q}_i)}{\dots} \phi^-(q_n) \phi(\mathbf{q}_i) \phi(k_1) \dots \phi(k_m)$$

SU(2)/1: new soft theorem

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$$x \sum_{i=1}^m \phi^+(\mathbf{k}_i) \dots \phi^+(p_n) \phi^-(q_1) \dots \phi^-(q_n) \phi(k_1) \overset{\phi(\mathbf{k}_i)}{\dots} \phi(k_m)$$

$$+y \sum_{i=1}^n \phi^+(p_1) \dots \phi^+(p_n) \phi^-(q_1) \overset{\phi(\mathbf{q}_i)}{\dots} \phi^-(q_n) \phi(\mathbf{q}_i) \phi(k_1) \dots \phi(k_m)$$

- IT WORKS!
- y = -x
- verified on amplitudes up to 7-pt
- can be proved from Lagrangian
- new generalization for the formal Lagrangian (work in progress)

Summary

- We have offered a new tool for effective field theories
- motivated by the amplitude methods employed for renormalizable theories
- analogy between gravity and soft scalar theories (Bonifacio et al. 'today)
- used for classification of scalar theories
- one new theory discovered: special galileon
- one exceptional theory for spin-1 particles: BI
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Thank you!